Summary sheet: Kinematics

Q3	Understand, use and derive the formulae for constant acceleration for motion in a straight line; extend to 2 dimensions using vectors
Q4	Use calculus in kinematics for motion in a straight line; extend to 2 dimensions using vectors

The constant acceleration formulae in two dimensions

Four of the constant acceleration equations (introduced at AS level for motion in a straight line) can be applied to motion in two dimensions by using vectors for displacement, velocity and acceleration.

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$\mathbf{s} = \tfrac{1}{2}(\mathbf{u} + \mathbf{v})t$$
$$\mathbf{s} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2$$
$$\mathbf{s} = \mathbf{v}t - \tfrac{1}{2}\mathbf{a}t^2$$

(The fifth equation, $v^2 = u^2 + 2as$, isn't covered here as the squared terms mean that it can't be converted to vector form so easily).

For example, if a particle with initial velocity $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ has displacement after 2 seconds of $\begin{pmatrix} 10 \\ -6 \end{pmatrix}$, you can use the vector equation $\mathbf{s} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2$:

$$\begin{pmatrix} 10 \\ -6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \times 2 + \tfrac{1}{2}\mathbf{a} \times 2^2$$
$$\begin{pmatrix} 10 \\ -6 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + 2\mathbf{a}$$
$$2\mathbf{a} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$
$$\mathbf{a} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

You can use the form $a\mathbf{i} + b\mathbf{j}$ for the vectors if you prefer, but column vectors are often easier because it's easy to do the calculations for each row.

Variable acceleration in two dimensions

At AS level, you used calculus to work with variable acceleration. These ideas can be extended to two dimensions using vectors.

Using differentiation
You can find the velocity vector by differentiating the position vector.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

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e.g. if the displacement $\mathbf{r}$ of a particle at time t is given by $\mathbf{r} = \begin{pmatrix} 3t^2 \\ 2t+1 \end{pmatrix}$

then the velocity $\mathbf{v}$ of the particle at time t is given by $\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{pmatrix} 6t \\ 2 \end{pmatrix}$

Similarly, you can find the acceleration vector by differentiating the velocity vector.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

e.g. if the velocity $\mathbf{v}$ of a particle at time t is given by $\mathbf{v} = \begin{pmatrix} -3t \\ 4\sqrt{t} \end{pmatrix}$

then the acceleration $\mathbf{a}$ of the particle at time t is given by $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} -3 \\ 2t^{-\frac{1}{2}} \end{pmatrix}$.

Using integration

Since integration is the reverse of differentiation, the results above can be reversed:

$$\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t$$

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t$$

e.g. if the acceleration $\mathbf{a}$ of a particle at time t is given by $a = \begin{pmatrix} 2 \\ 4t-1 \end{pmatrix}$, and initially the particle is at the

origin ($\mathbf{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$) and has velocity given by $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ms^{-1}:

Notice that the constant of integration is a vector

then $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int \begin{pmatrix} 2 \\ 4t-1 \end{pmatrix} \mathrm{d}t = \begin{pmatrix} 2t \\ 2t^2 - t \end{pmatrix} + \mathrm{c}$

and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ when $t = 0$, so $\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and therefore $\mathbf{v} = \begin{pmatrix} 2t+1 \\ 2t^2 - t - 2 \end{pmatrix}$

Also $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int \begin{pmatrix} 2t+1 \\ 2t^2 - t - 2 \end{pmatrix} \mathrm{d}t = \begin{pmatrix} t^2 + t \\ \frac{2}{3}t^3 - \frac{1}{2}t^2 - 2t \end{pmatrix} + \mathbf{k}$

and $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when $t = 0$, so $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and therefore $\mathbf{r} = \begin{pmatrix} t^2 + t \\ \frac{2}{3}t^3 - \frac{1}{2}t^2 - 2t \end{pmatrix}$

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