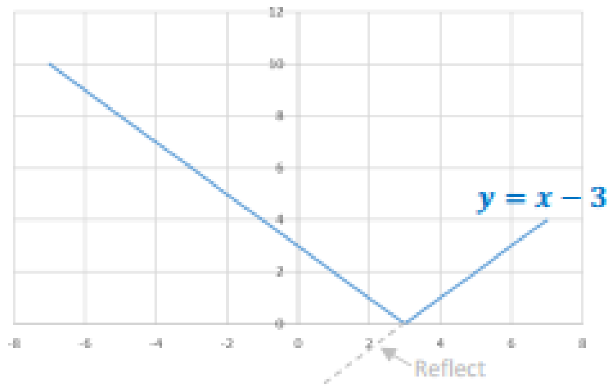
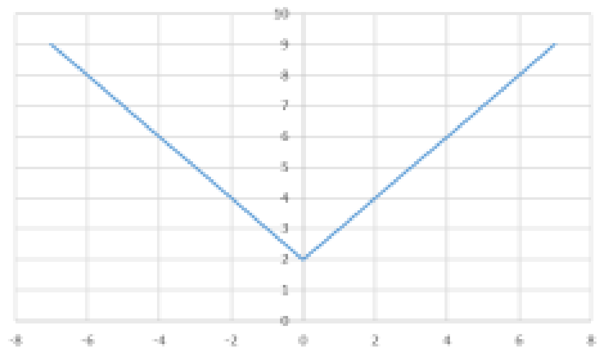
The modulus of a linear function

Remember that the modulus of a number means the absolute value (or "ignore the sign" because it is always positive). E.g. $|-5| = 5$

Sketching modulus graphs:

Sketch the graph without the modulus sign and then reflect the appropriate part.

Example:	What to do:	The graph:
$y = \|x - 3\|$	• Sketch $y = x - 3$ • The answer (y) cannot be negative so reflect the negative part in the x-axis.	
$y = \|x\| + 2$	• Sketch $y = x$ • Reflect the negative part in the x-axis • Translate the graph up 2.	

Functions

Remember that when dealing with function notation you can substitute the x for whatever you are trying to find, no matter how complicated.

e.g. if $f(x) = 5x - 3$, find $f(2)$ and $f(x^2 + 1)$

$$f(2) = 5(2) - 3 = 7$$

$$f(x^2 + 1) = 5(x^2 + 1) - 3 = 5x^2 + 2$$

Composite functions

Always apply the inside function first. e.g. to find $\text{fg}(x)$, find $\text{g}(x)$ first then substitute your answer into $\text{f}(x)$ to find $\text{f}(answer)$

e.g. If $\text{f}(x) = \frac{3}{x-2}$ and $\text{g}(x) = 2x - 4$ find $\text{fg}(7)$ and $\text{fg}(x)$

$\text{g}(7) = 2(7) - 4 = 10$ $\qquad$ $\text{f}(10) = \frac{3}{10-2} = \frac{3}{8}$

$\text{g}(x) = 2x - 4$ $\qquad$ $\text{f}(2x-4) = \frac{3}{2x-4-2} = \frac{3}{2x-6}$

Inverse functions

The inverse of $\text{f}(x)$ is written $\text{f}^{-1}(x)$. Remember that a function and its inverse 'undo' each other, so if you applied a function and then applied it's inverse you would be back where you started, i.e. $\text{ff}^{-1}(x) = x$ and $\text{f}^{-1}\text{f}(x) = x$

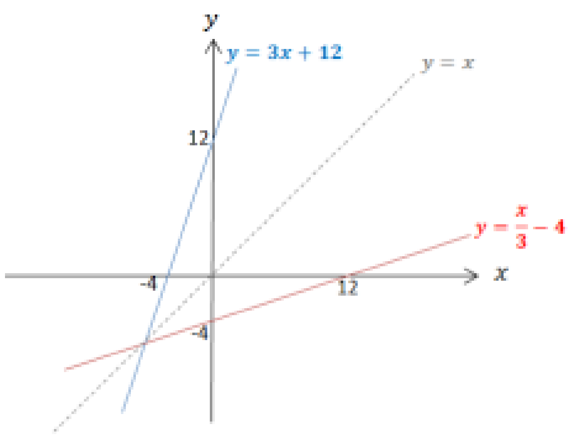
To find an inverse function:

Write as $y =$ Swap x and y Rearrange to make y the subject

e.g. if $\text{f}(x) = 3x + 12$ find $\text{f}^{-1}(x)$.

Write as $y =$	$y = 3x + 12$	
Swap x and y	$x = 3y + 12$	
Rearrange to make y the subject	$3y = x - 12$	
	$y = \frac{x}{3} - 4$	$\therefore \text{f}^{-1}(x) = \frac{x}{3} - 4$

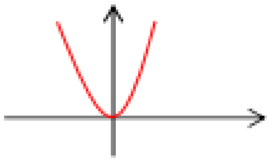
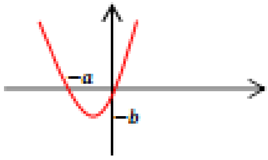
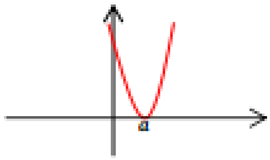
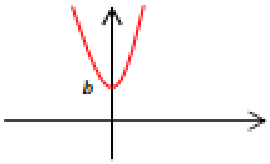
You can sketch the graphs of $\text{f}(x)$ and $\text{f}^{-1}(x)$ and you will see that they reflect in the line $y = x$.



Transformations

See the AS summary sheet for single transformations. Transformations can also be combined, you need to think about what each part of the transformation would do then apply both (all) of them.

The following gives examples of some combination graph transformations:

e.g. graph: $y = x^2$	Original	
$y = (x + a)^2 - b$	Translates left a and down b	
$y = b(x - a)^2$	Translates right a then stretches in the y-direction with factor b	
$y = ax^2 + b$	Stretches in the y-direction with factor a then translates up b	

Remember that these transformations work for all graphs, $y = x^2$ is just an example.
Try different graphs and combinations yourself to see what happens, and try replacing a and b with numbers to see how they affect your graphs.

Sequences and series

A **sequence** is a set of numbers in a given order – they could form a pattern.
A **series** is the sum of the consecutive terms of a sequence.

Some types of sequences

Increasing:	Each number is bigger than the previous, i.e. $a_{n+1} > a_n$
Decreasing:	Each number is smaller than the previous, i.e. $a_{n+1} < a_n$
Periodic:	The sequence repeats (e.g. 1, 4, 7, 1, 4, 7, 1, 4, 7,............)
Finite:	Has a first and a last term (it comes to an end).
Infinite:	No last term – continues to infinity.
Convergent:	Approaches a limit. You would use limit notation to denote the value it converges to. e.g. for the sequence 2.9, 2.99, 2.999, 2.9999, $\lim_{n\to\infty}(a_n) = 3$
Divergent:	Doesn't have a limit.

Some meanings

∑ (pronounced "sigma"): Means "the sum of"

e.g. $\sum_{k=1}^{k=4} 3^k$ means sum $3^1 + 3^2 + 3^3 + 3^4 = 120$

$\mathbf{x_{n+1} = f(x_n)}$: Means "the next number in the sequence (x_{n+1}) is the function of the previous number $\left(f(x_n)\right)$"

e.g. Write down the first five terms of the sequence, $x_{n+1} = 3x_n + 4$ given that $x_0 = 1$
(i.e. to find the next term do 3 x previous term + 4)
$x_0 = 1$
$x_1 = 3(1) + 4 = 7$
$x_2 = 3(7) + 4 = 25$
$x_3 = 3(25) + 4 = 79$
$x_4 = 3(79) + 4 = 241$

The sequence is: 1, 7, 25, 79, 241

Y13 Pure Chapter 3—Sequences and Series

Arithmetic sequences & series

An **arithmetic sequence** (sometimes called an arithmetic progression (AP)) has a **common difference** (i.e. the difference between consecutive terms is the same). E.g. 2, 5, 8, 11, 14, 17 is an arithmetic sequence with a common difference of 3. You need to be able to use the formulae to find any term in the sequence and also to sum together the terms of a sequence.

This can be used if you know the first and the last term.

Find the k_{th} term:	$a_k = a + (k-1)d$

Sum together the 1st n terms:	$S_n = \frac{1}{2}n[2a + (n-1)d]$	OR	$S_n = \frac{1}{2}n(a+L)$

Where: a is 1st term, d is common difference and L is last term

e.g. for the sequence: 3, 7, 11, 15, 19....... Find the 12th term and sum of the first 8 terms.

You know that $a = 3$ and $d = 4$

$$a_{12} = 3 + (12-1)4 = 47$$
$$S_8 = \frac{1}{2} \times 8(2 \times 3 + (8-1)4) = 136$$

Geometric sequences & series

A **geometric sequence** (sometimes called a geometric progression (GP)) has a **common ratio** (i.e. you find the next term by multiplying the previous term by a fixed number) E.g. 2, 6, 18, 54, 162 is a geometric sequence with a common ratio of 3. You need to be able to use the formulae to find any term in the sequence and also to sum together the terms of the sequence.

Use whichever is easiest (e.g. the first one if $r < 1$ and the second one if $r > 1$)

Find the k_{th} term:	$a_k = ar^{k-1}$

Sum together the 1st n terms:	$S_n = \frac{a(1-r^n)}{1-r}$	OR	$S_n = \frac{a(r^n-1)}{r-1}$

Sum to infinity:	$S_\infty = \frac{a}{1-r}$	**For $-1 < r < 1$** **i.e.: $\lvert r \rvert < 1$**

N.B. You can only sum to infinity if $|r| < 1$ because if $|r| > 1$ the terms will get larger and larger and the series will not converge.

e.g. for the sequence: 1, 2, 4, 8, 16, 32 .... Find the 15th term and sum of the first 10 terms.

You know that $a = 1$ and $r = 2$

$$a_{15} = 1 \times 2^{14} = 16384$$
$$S_{10} = \frac{1(2^{10}-1)}{2-1} = 1023$$

The binomial expansion

You have already seen how to use the binomial expansion for $(a+b)^n$ where n is a positive integer (see AS summary sheet). In this case there are a finite number of terms.

However, when using the binomial expansion where n is NOT a positive integer, there will be an infinite number of terms. This means that the **binomial expansion can only be used when $-1 < x < 1$** because then x^n will decrease as n increases and there will be a limit.

Remember:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots$$

If you take note of the pattern it will make this easier to remember (multiply **2** n terms and use x to the power of **2** on the top, and **2**! on the bottom, multiply **3** n terms and use x to the power of **3** on the top, and **3**! on the bottom etc).

e.g. Expand $(1+x)^{-4}$ up to the x^3 term

$n = -4$ — 3 brackets — Power of 3 — 3!

$$(1+x)^{-4} = 1 + (-4)x + \frac{(-4)(-5)x^2}{2!} + \frac{(-4)(-5)(-6)x^3}{3!} \ldots\ldots$$

$$= 1 - 4x + 10x^2 - 20x^3$$

Valid when $-1 < x < 1$

Different 2nd term

If the 2nd term in the bracket is different you can still expand as above but you need to be careful:

- Remember to do the whole 2nd term to the power
- Be careful if there is a negative number; always do the whole term to the power.

e.g. Expand $\sqrt[4]{1-3x}$ up to the x^2 term

First rewrite the question as $(1-3x)^{\frac{1}{4}}$ (and remember that you will be using $-3x$ instead of x).

$$(1-3x)^{\frac{1}{4}} = 1 + \frac{1}{4}(-3x) + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)(-3x)^2}{2!}$$

$$= 1 - \frac{3}{4}x - \frac{27x^2}{32}$$

$n = \frac{1}{4}$

Replace x with $-3x$

Valid when $-1 < 3x < 1$ $\therefore$ $-\frac{1}{3} < x < \frac{1}{3}$ can be written as $|x| < \frac{1}{3}$

Different 1st term

If the 1st term in the bracket is not 1, you can still use the expansion. You will need to take out a factor to make the 1st term=1.

Remember: $$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$

e.g. Expand $(3-x)^{-3}$ up to the x^3 term

First rewrite the question as: $3^{-3}\left(1-\frac{x}{3}\right)^{-3}$

$n = -3$

Replace x with $-\frac{x}{3}$

$$3^{-3} = \frac{1}{27}$$

$$\left(1-\frac{x}{3}\right)^{-3} = 1 + (-3)\left(-\frac{x}{3}\right) + \frac{(-3)(-4)\left(-\frac{x}{3}\right)^2}{2!} + \frac{(-3)(-4)(-5)\left(-\frac{x}{3}\right)^3}{3!}$$

$$3^{-3}\left(1-\frac{x}{3}\right)^{-3} = \frac{1}{27}\left(1 + x + \frac{2x^2}{3} + \frac{10x^3}{27}\right)$$

Valid when $-1 < \frac{1}{3}x < 1$ $\therefore$ $-3 < x < 3$ can be written as $|x| < 3$

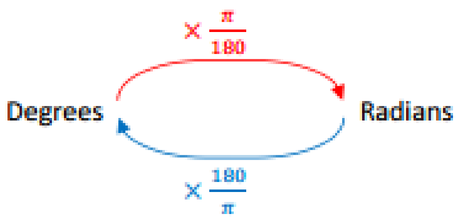
Radians

Radians can be used (instead of degrees) to measure angles, and they can sometimes make calculations easier.

Remember:

$180° = \pi$ rads
$360° = 2\pi$ rads

Converting between degrees and radians:



There are some common angles that you should know and remember (these should be expressed as fractions of π instead of as decimals). Remembering that $180° = \pi$ rads should help you to remember:

$30° = \frac{\pi}{6}$ radians	$45° = \frac{\pi}{4}$ radians	$60° = \frac{\pi}{3}$ radians	$90° = \frac{\pi}{2}$ radians

Arc length

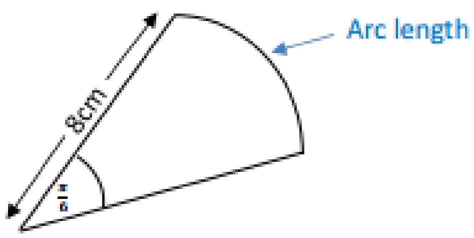
Arc length = $r\theta$

Area of a sector

Area of a sector = $\frac{1}{2}r^2\theta$

(where r is the radius of the circle and θ is the angle at the centre (in radians))

e.g. find the arc length and area for the following sector:



Arc length = $8 \times \frac{\pi}{6} = \frac{4\pi}{3}$ (4.2)

Area = $\frac{1}{2} \times 8^2 \times \frac{\pi}{6} = \frac{16\pi}{3}$ (16.8)

Standard small angle approximations of sine, cosine and tangent

$\sin\theta \approx \theta$	$\cos\theta \approx 1 - \frac{\theta^2}{2}$	$\tan\theta \approx \theta$

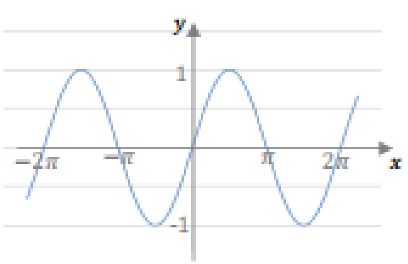
For small angles where θ is the angle in radians.

Some common angles

You should know the exact values of sin, cos and tan for the following angles (and their multiples):

$\theta =$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined	0

The graphs of $\sin x$, $\cos x$ and $\tan x$

<table>
<tr><th>$y = \sin x$</th><th>$y = \cos x$</th><th>$y = \tan x$</th></tr>
<tr>
<td>
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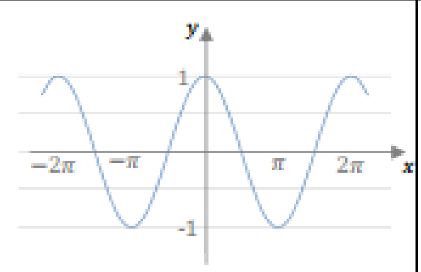
<td>
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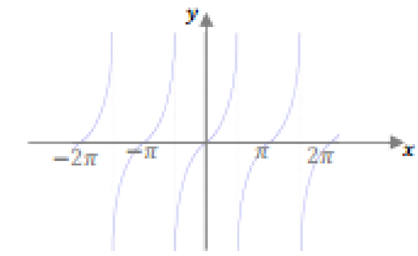
<td>
</td>
</tr>
<tr>
<td>• Period = 2π
• Rotational symmetry about the origin
• Lies between -1 and 1
• Line of symmetry at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$</td>
<td>• Period = 2π
• Line of symmetry at y-axis
• Lies between -1 and 1
• $y = \cos x$ is the same as $y = \sin x$ shifted left by $\frac{\pi}{2}$</td>
<td>• Period = π
• Rotational symmetry about the origin
• Lies between $-\infty$ and ∞
• Asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ etc</td>
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Rigid bodies

In some mechanics problems, objects can be modelled as particles. This means that they are treated as a single point with no size.

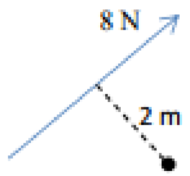
Some objects, such as beams or ladders, cannot realistically be modelled as particles. Instead they can be modelled as rigid bodies: they have size, but they can be treated as if they cannot be bent or distorted.

When forces act on a rigid body, they can cause the body to rotate about a point.

Moments

The moment of a force about a fixed point is the turning effect of the force.

The moment of a force F about a point is given by Fd, where d is the perpendicular distance of the line of the force from the point. Moments are measured in Nm (Newton metres).

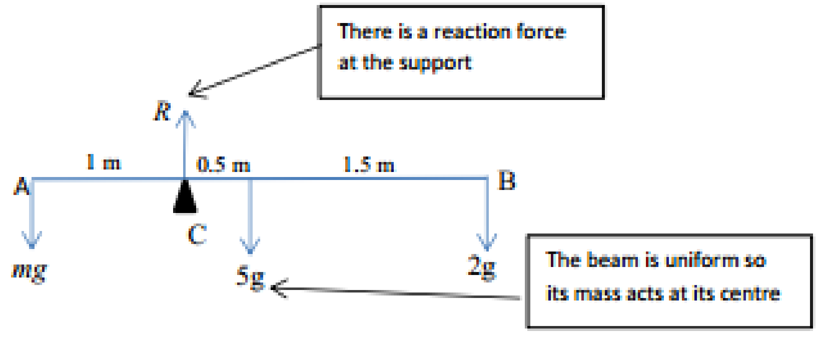


The moment of the 8 N force about the point O is given by $8 \times 2 = 16$ Nm clockwise.

Equilibrium

For a rigid body to be in equilibrium, there must be zero resultant force (so that there is no acceleration), and also there must be zero total moment (so that there is no turning effect).

This diagram shows a uniform beam AB of length 3 m and mass 5 kg resting on a support 1 m from A. There is a mass of 2g at end B. Another mass is placed at end A so that the beam is in equilibrium.



In this situation, it is easiest to take moments about point C, because the reaction force has zero moment at this point.

Clockwise moment of the weight about C

Clockwise moment of the 2 kg mass about C

Anticlockwise moment of the m kg mass about C

Total moment is 0 since the beam is in equilibrium

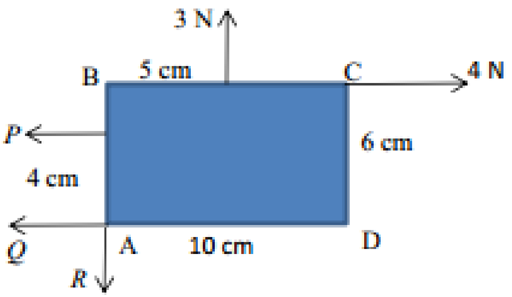
$$5g \times 0.5 + 2g \times 2 - mg \times 1 = 0$$
$$mg = 6.5g$$
$$m = 6.5$$

You can also find the reaction force at the support by looking at the vertical forces.

$$R - 5g - 2g - 6.5g = 0$$
$$R = 13.5g = 132.3\text{ N}$$

The next example shows forces on a rectangular lamina. A lamina is a rigid body which is modelled as having zero thickness.

The lamina is in equilibrium. What are the forces P, Q and R?



Vertical forces: $3 - R = 0$ so $R = 3$ N

Taking moments about A: $4P + 3\times5 - 4\times6 = 0$

$$4P = 24 - 15$$
$$P = 2.25\text{ N}$$

Horizontal forces: $4 - P - Q = 0$

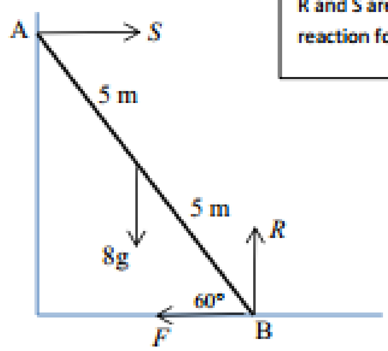
$$Q = 4 - 2.25 = 1.75\text{ N}$$

Moments and resolving forces

Sometimes you need to resolve forces in order to take a moment.

The diagram shows a uniform beam AB of length 10 m and weight 8 kg leaning with end A against a smooth wall and end B on rough ground.

What frictional force is needed at B to stop the beam from slipping?



R and S are reaction forces

The beam is uniform so its weight acts at the centre

F is the frictional force

Taking moments about A: each of the forces is resolved into a component along the beam (which has no moment about A) and a component perpendicular to the beam. The components of the forces perpendicular to the beam are $F\sin60°$, $R\cos60°$, and $8g\cos60°$.

So the 'moments about A' equation is: $F\sin60°\times10 + 8g\cos60°\times5 - R\cos60°\times10 = 0$

Resolving vertically gives $R = 8g$, so this gives $F = 22.6$ N.